

PRODUCTS IN HOPF-CYCLIC COHOMOLOGY

ATABEY KAYGUN

1. INTRODUCTION

It is a truism to say that the study of Hopf algebras provides an unified framework for a wide range of algebras such as group rings, enveloping algebras of Lie algebras and quantum groups. Beyond this unified framework, Hopf algebras also provide the true symmetries of noncommutative spaces. Yet, as it is observed in [17] and [14] the notion of ‘symmetry of a noncommutative space’ is not as straightforward as its classical counterpart in that there are fundamentally different types of ‘equivariances’ for noncommutative spaces. In this article, we will investigate cohomological consequences of these different types of Hopf equivariances under various scenarios.

Hopf-cyclic (co)homology and equivariant cyclic cohomology of module (co)algebras, in their most general form as defined in [15] and [8], can now be described via appropriate bivariant cohomology theories [16]. These bivariant cohomology theories provide us the right vocabulary and the right kind of tools to investigate various kinds of products and pairings in the Hopf-cyclic and equivariant cyclic settings. We already established in [16] that equivariant cyclic (co)homology groups of H -module (co)algebras are graded modules over the graded algebra $\text{Ext}_H^*(k, k)$ where H is the underlying Hopf algebra. Not only is this the right kind of equivariance for an equivariant cohomology theory but also is a very useful computational tool in obtaining new equivariant cyclic classes from old ones by using the action. However, Hopf-cyclic cohomology lacks such an action precisely because it is, by design, oblivious to the cohomology of the underlying Hopf algebra viewed as an algebra or a coalgebra depending on the type of the symmetry at hand. To improve the situation, in Theorem 2.8 we develop a pairing, similar to one of the cup products obtained in [18], of the form

$$\smile: HC_{\text{Hopf}}^p(C, M) \otimes HC_{\text{Hopf}}^q(A, M) \rightarrow HC^{p+q}(A)$$

for an H -module coalgebra C acting equivariantly on a H -module algebra A (Definition 2.2) where M is an arbitrary coefficient module/comodule. The most striking corollary to this pairing is Theorem 2.10 where we recover Connes-Moscovici characteristic map [3, Section VIII, Proposition 1] by letting $q = 0$, $C = H$ and letting $M = k_{(\sigma, \delta)}$ a coefficient module coming from a modular pair in involution. The ideas instrumental in constructing this pairing are the derived functor interpretation of Hopf-cyclic and equivariant cyclic cohomology [16], and the Yoneda interpretation of Ext-groups [24]. We use the same ideas with little modification in Theorem 2.11 to obtain a completely new pairing in equivariant cyclic cohomology of the form

$$\smile: HC_H^p(C, M) \otimes HC_H^q(A, M) \rightarrow \bigoplus_{r=0}^{p+q} HC^{p+q-r}(A) \otimes \text{Ext}_H^r(k, k)$$

where, as before, C is an H -module coalgebra acting equivariantly on an H -module algebra A , and M is an arbitrary coefficient module/comodule.

Once we couple equivariant/Hopf-cyclic cohomology and the Yoneda product, we obtain other types of products and pairings by imposing various conditions on the underlying Hopf algebra, the coefficient modules and (co)module (co)algebras involved

$$HC_{\text{Hopf}}^p(A, M) \otimes HC_{\text{Hopf}}^q(A', M') \rightarrow HC_{\text{Hopf}}^{p+q}(A \otimes A', M \square^H M') \quad \text{Theorem 3.2}$$

$$HC_H^p(A, M) \otimes HC_H^q(A', M') \rightarrow HC_H^{p+q}(A \otimes A', M \square^H M') \quad \text{Theorem 3.2}$$

$$HC_{\text{Hopf}}^{\vee, p}(Z, M) \otimes HC_{\text{Hopf}}^{\vee, q}(Z', M') \rightarrow HC_{\text{Hopf}}^{\vee, p+q}(Z \otimes Z', M \otimes_H M') \quad \text{Theorem 3.5}$$

$$HC_{\text{Hopf}}^p(A, M) \otimes HC_{\text{Hopf}}^q(B, M) \rightarrow HC^{p+q}(A \rtimes B) \quad \text{Theorem 4.4}$$

$$HC^p(Z \ltimes C) \otimes HC_{\text{Hopf}}^{\vee, q}(Z, M) \rightarrow HC_{\text{Hopf}}^{p+q}(C, M) \quad \text{Theorem 4.7}$$

where HC^* , HC_{Hopf}^* , HC_H^* and $HC_{\text{Hopf}}^{\vee, *}$ are ordinary cyclic, Hopf-cyclic, equivariant cyclic and dual Hopf cyclic cohomology functors respectively.

The last section of our paper is devoted to ramifications of an interesting technical problem in cyclic cohomology which manifests itself here in the context of pairings in Hopf-cyclic (co)homology. It is clear by now that there are essentially two different homotopical frameworks for the category cyclic modules: (i) Connes' derived category of cyclic modules [1] and (ii) Cuntz-Quillen formalism of homotopy category of towers of super complexes [5] which is equivalent to the derived category of mixed complexes [12] and the derived category of S -modules [13] by [22]. One can see the difference in the simple fact that for a cyclic module X_\bullet , the derived functors $\text{Ext}_\Lambda^*(k_\bullet^\vee, X_\bullet)$ in the category of cyclic modules compute the dual cyclic homology of X_\bullet while the derived functors $\mathbf{Ext}_\mathcal{M}^*(\mathcal{B}_*(k_\bullet^\vee), \mathcal{B}_*(X_\bullet))$ in the category of mixed complexes compute the negative cyclic homology of X_\bullet [11, Theorem 2.3]. Here k_\bullet is the cocyclic module of the ground field viewed as a coalgebra, the dual cyclic homology of X_\bullet is the cyclic cohomology of X_\bullet^\vee , the cyclic dual of a (co)cyclic module X_\bullet defined by using Connes' duality functor [1], and $\mathcal{B}_*(Z_\bullet)$ is the mixed complex of a (co)cyclic module Z_\bullet . The main result of this last section is Theorem 5.4 where we prove replacing the derived category cyclic modules by the derived category of mixed complexes, or S -modules, or the homotopy category of towers of super complexes in Theorem 2.8, Theorem 2.11 and Theorem 3.2 will not change the pairings we already defined.

In this article k will denote an arbitrary field. We make no assumption about the characteristic of k . We will use H to denote a bialgebra, or a Hopf algebra with an invertible antipode over k , whenever necessary. All tensor products, unless otherwise explicitly stated, are over k .

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2. EQUIVARIANT ACTIONS OF COALGEBRAS ON ALGEBRAS

In this section, A will denote a unital associative left H -module algebra and C will denote a counital coassociative left H -module coalgebra. Explicitly, one has

$$h(a_1 a_2) = (h_{(1)} a_1)(h_{(2)} a_2) \quad \text{and} \quad (hc)_{(1)} \otimes (hc)_{(2)} = h_{(1)} c_{(1)} \otimes h_{(2)} c_{(2)}$$

for any $a_1, a_2 \in A$, $c \in C$ and $h \in H$. We also assume

$$h(1_A) = \varepsilon(h)1_A \quad \text{and} \quad \varepsilon(hc) = \varepsilon(h)\varepsilon(c)$$

for any $c \in C$ and $h \in H$. We will use M to denote an arbitrary H -module/comodule with no assumption on the interaction between the H -module and H -comodule structures on M .

Given two morphisms $f_1, f_2 \in \text{Hom}_k(C, A)$ we define their convolution product as

$$(f_1 * f_2)(c) := f_1(c_{(1)})f_2(c_{(2)})$$

for any $c \in C$. This binary operation on $\text{Hom}_k(C, A)$ is an associative product. The unit for this algebra is $\eta(c) := \varepsilon(c)1_A$. The proof of the following lemma is routine.

Lemma 2.1. *There exists a morphism of algebras of the form $\alpha: A \rightarrow \text{Hom}_k(C, A)$ if and only if one has a pairing $\phi: C \otimes A \rightarrow A$ which satisfies*

$$\phi(c, a_1 a_2) = \phi(c_{(1)}, a_1)\phi(c_{(2)}, a_2) \quad \text{and} \quad \phi(c, 1) = \varepsilon(c)1_A$$

for any $c \in C$, $a, a_1, a_2 \in A$ and $h \in H$.

Definition 2.2. If one has a pairing between a module coalgebra C and a module algebra A as described in Lemma 2.1 then C is said to act on A . Such an action is going to be called equivariant if one also has

$$h\phi(c, a) = \phi(hc, a)$$

for any $a \in A$, $h \in H$ and $c \in C$.

One can observe that an H -module coalgebra C acts on an H -module algebra A equivariantly if and only if the canonical morphism of algebras $\alpha: A \rightarrow \text{Hom}_k(C, A)$ factors through the inclusion $\text{Hom}_H(C, A) \subseteq \text{Hom}_k(C, A)$ [18].

Definition 2.3. If X_\bullet is a (para-)cocyclic k -module and Y_\bullet is a (para-)cyclic k -module then the graded module

$$\text{diagHom}_k(X_\bullet, Y_\bullet) := \bigoplus_n \text{Hom}_k(X_n, Y_n)$$

carries a (para-)cyclic module structure defined as

$$\begin{aligned} (\partial_j f)(x_{n-1}) &:= \partial_j^Y f(\partial_j^X(x_{n-1})) \quad 0 \leq j \leq n \\ (\sigma_j f_n)(x_{n+1}) &:= \sigma_j^Y f(\sigma_j^X(x_{n+1})) \quad 0 \leq j \leq n \\ (\tau_n f_n)(x_n) &:= \tau_{n,Y} f(\tau_{n,X}(x_n)) \end{aligned}$$

for any $f \in \text{Hom}_k(X_n, Y_n)$, $x_i \in X_i$ for $i = n-1, n, n+1$.

We will use $\text{Cyc}_\bullet(X)$ to denote the classical cyclic k -module of an (co)associative (co)unital k -(co)algebra X . Also, we will use k_\bullet to denote $\text{Cyc}_\bullet(k^c)$ the *cocyclic* k -module of the ground field viewed as a coalgebra and k_\bullet^\vee to denote $\text{Cyc}_\bullet(k)$ the *cyclic* k -module of the ground field k viewed as an algebra. Note that in this case k_\bullet^\vee is actually the cyclic dual of the cocyclic k -module k_\bullet in the sense of [1, Lemme 1]. In general, if X_\bullet is an arbitrary (co)cyclic k -module, X_\bullet^\vee will denote its cyclic dual. $T_\bullet(C, M)$ and $T_\bullet(A, M)$ are going

to denote the para-(co)cyclic complex (also referred as “the cover complex”) of the H -module coalgebra C and H -module algebra A with coefficients in a H -module/comodule M respectively [16].

Here we recall the para-(co)cyclic structure morphisms on both $T_\bullet(C, M)$ and $T_\bullet(A, M)$ from [16]. The modules are defined as

$$T_n(C, M) := C^{\otimes n+1} \otimes M \quad \text{and} \quad T_n(A, M) := A^{\otimes n+1} \otimes M$$

We define the structure morphisms on $T_\bullet(C, M)$ by

$$\begin{aligned} \partial_0(c^0 \otimes \cdots \otimes c^n \otimes m) &:= c_{(1)}^0 \otimes c_{(2)}^0 \otimes c^1 \otimes \cdots \otimes c^n \otimes m \\ \sigma_0(c^0 \otimes \cdots \otimes c^n \otimes m) &:= c^0 \otimes \varepsilon(c^1) \otimes c^2 \otimes \cdots \otimes c^n \otimes m \\ \tau_n(c^0 \otimes \cdots \otimes c^n \otimes m) &:= c^1 \otimes \cdots \otimes c^n \otimes m_{(-1)} c^0 \otimes m_{(0)} \end{aligned}$$

Then we define $\partial_j := \tau_{n+1}^{-j} \partial_0 \tau_n^j$ and $\sigma_i := \tau_{n-1}^{-j} \sigma_0 \tau_n^j$. Similarly, we let

$$\begin{aligned} \partial_0(a_0 \otimes \cdots \otimes a_n \otimes m) &:= a_0 a_1 \otimes a_2 \otimes a_n \otimes m \\ \sigma_0(a_0 \otimes \cdots \otimes a_n \otimes m) &:= a_0 \otimes 1_A \otimes a_1 \otimes \cdots \otimes a_n \otimes m \\ \tau_n(a_0 \otimes \cdots \otimes a_n \otimes m) &:= S^{-1}(m_{(-1)}) a_n \otimes a_0 \otimes \cdots \otimes a_{n-1} \otimes m_{(0)} \end{aligned}$$

Then we define $\partial_j := \tau_{n-1}^j \partial_0 \tau_n^{-j}$ and $\sigma_j := \tau_{n+1}^j \sigma_0 \tau_n^{-j}$.

Proposition 2.4. *Assume C acts on A equivariantly and let M be an arbitrary H -module/comodule. Let us define*

$$(2.1) \quad \alpha_n(a_0 \otimes \cdots \otimes a_n)(c^0 \otimes \cdots \otimes c^n \otimes m) := \phi(c^0, a_0) \otimes \cdots \otimes \phi(c^n, a_n) \otimes m$$

for any $n \geq 0$, $m \in M$, $a_i \in A$, $c^i \in C$ for $i = 0, \dots, n$. Then α_\bullet defines a morphism of para-cyclic k -modules of the form

$$\alpha_\bullet : Cyc_\bullet(A) \rightarrow diagHom_H(T_\bullet(C, M), T_\bullet(A, M))$$

Proof. It is easy to observe that α_n is H -linear since the action of C on A is equivariant. Let us check if α_\bullet now defines a morphism of para-cyclic modules:

$$\begin{aligned} (2.2) \quad \alpha_{n-1}(\partial_0^A(a_0 \otimes \cdots \otimes a_n))(c^0 \otimes \cdots \otimes c^{n-1} \otimes m) \\ = \phi(c^0, a_0 a_1) \otimes \phi(c^1, a_2) \otimes \cdots \otimes \phi(c^{n-1}, a_n) \otimes m \\ = \phi(c_{(1)}^0, a_0) \phi(c_{(2)}^0, a_1) \otimes \phi(c^1, a_2) \otimes \cdots \otimes \phi(c^{n-1}, a_n) \otimes m \\ = \partial_0^{(A, M)} \alpha_n(a_0 \otimes \cdots \otimes a_n) \partial_0^{(C, M)}(c^0 \otimes \cdots \otimes c^{n-1} \otimes m) \end{aligned}$$

$$\begin{aligned} (2.3) \quad \alpha_{n+1}(\sigma_0^A(a_0 \otimes \cdots \otimes a_n))(c^0 \otimes \cdots \otimes c^{n+1} \otimes m) \\ = \phi(c^0, a_0) \otimes \varepsilon(c_1) 1_A \otimes \phi(c^2, a_1) \otimes \cdots \otimes \phi(c^{n+1}, a_n) \otimes m \\ = \sigma_0^{(A, M)} \alpha_{n+1}(a_0 \otimes \cdots \otimes a_n) \sigma_0^{(C, M)}(c^0 \otimes \cdots \otimes c^{n+1} \otimes m) \end{aligned}$$

$$\begin{aligned} (2.4) \quad \alpha_n(\tau_n(a_0 \otimes \cdots \otimes a_n))(c_0 \otimes \cdots \otimes c_n \otimes m) \\ = \phi(c^0, a_n) \otimes \phi(c^1, a_0) \otimes \cdots \otimes \phi(c^n, a_{n-1}) \otimes m \\ = S^{-1}(m_{(-1)}) \phi(m_{(-2)} c^0, a_n) \otimes \phi(c^1, a_0) \otimes \cdots \otimes \phi(c^n, a_{n-1}) \otimes m_{(0)} \\ = \tau_{n, (A, M)}(\phi(c^1, a_0) \otimes \cdots \otimes \phi(c^n, a_{n-1}) \otimes \phi(m_{(-1)} c^0, a_n) \otimes m_{(0)}) \\ = \tau_{n, (A, M)} \alpha_n(a_0 \otimes \cdots \otimes a_n) \tau_{(C, M)}(c_0 \otimes \cdots \otimes c_n \otimes m) \end{aligned}$$

for any $c^i \in C$, $a_i \in A$, $m \in M$. \square

Remark 2.5. Now, we will recall few relevant definitions from [16].

Let $J_\bullet(C, M)$ be the smallest para-cocyclic k -submodule and graded H -submodule (but not necessarily the para-cocyclic H -subcomodule) of $T_\bullet(C, M)$ generated by elements of the form $[L_h, \tau_n^i](\Psi) + (\tau_n^{n+1} - id_n)(\Phi)$ where $\Psi, \Phi \in T_n(C, M)$, $i \in \mathbb{Z}$ and L_h is the graded k -module endomorphism of $T_\bullet(C, M)$ coming from the left diagonal action of $h \in H$ on $T_n(C, M)$ for each $n \geq 0$. One can similarly define $J_\bullet(A, M)$.

We define $Q_\bullet(C, M) := T_\bullet(C, M)/J_\bullet(C, M)$. One can see that $Q_\bullet(C, M)$ is a cocyclic H -module. Similarly $Q_\bullet(A, M) := T_\bullet(A, M)/J_\bullet(A, M)$ is a cyclic H -module. This cocyclic (resp. cyclic) H -module is called the H -equivariant cocyclic (resp. cyclic) module of the pair (C, M) (resp. (A, M)). The cyclic cohomology of the (co)cyclic H -modules $Q_\bullet(C, M)$ and $Q_\bullet(A, M)$ will be denoted by $HC_H^*(C, M)$ and $HC_H^*(A, M)$ respectively.

We define $C_\bullet(C, M) := k \otimes_H Q_\bullet(C, M)$. One can see that $C_\bullet(C, M)$ is a cocyclic k -module. Similarly $C_\bullet(A, M) := k \otimes_H Q_\bullet(A, M)$ is a cyclic k -module. This cocyclic (resp. cyclic) k -module is called the Hopf-cocyclic (resp. Hopf-cyclic) module of the pair (C, M) (resp. (A, M)). The cyclic cohomology of the (co)cyclic k -modules $C_\bullet(C, M)$ and $C_\bullet(A, M)$ will be denoted by $HC_{\text{Hopf}}^*(C, M)$ and $HC_{\text{Hopf}}^*(A, M)$ respectively.

Lemma 2.6. *For any $n \geq 0$ and $a_i \in A$ for $0 \leq i \leq n$ the restriction of the H -linear morphism $\alpha_n(a_0 \otimes \cdots \otimes a_n)$ in $\text{Hom}_k(T_n(A, M), T_n(C, M))$ to $J_n(C, M)$ is a morphism in $\text{Hom}_H(J_n(C, M), J_n(A, M))$.*

Proof. It is sufficient to prove that for every $n \geq 0$, $a_i \in A$ with $0 \leq i \leq n$ the morphism $\alpha_n(a_0 \otimes \cdots \otimes a_n)$ maps the elements of the form $[L_h, \tau_n^i](\Psi) + (\tau_n^{n+1} - id_n)(\Phi)$ to elements of the same form. We know that each $\alpha_n(a_0 \otimes \cdots \otimes a_n)$ is H -linear. So, we can reduce the proof to verification of the following string of equalities:

$$\begin{aligned} & \alpha_n(a_0 \otimes \cdots \otimes a_n) \tau_{n, (C, M)}(c_0 \otimes \cdots \otimes c_n \otimes m) \\ &= \alpha_n(a_0 \otimes \cdots \otimes a_n) (c^1 \otimes \cdots \otimes c^n \otimes m_{(-1)} c^0 \otimes m_{(0)}) \\ &= \phi(c^1, a_0) \otimes \cdots \otimes \phi(c^n, a_{n-1}) \otimes m_{(-1)} \phi(c^0, a_n) \otimes m_{(0)} \\ &= \tau_{n, (A, M)}^{-1} (\phi(c^0, a_n) \otimes \phi(c^1, a_0) \otimes \cdots \otimes \phi(c^n, a_{n-1}) \otimes m) \end{aligned}$$

for $m \in M$, $a_i \in A$, $c^i \in C$ with $0 \leq i \leq n$. \square

Proposition 2.7. α_\bullet can be extended to morphism of cyclic k -modules of the form

$$\alpha_\bullet: \text{Cyc}_\bullet(A) \rightarrow \text{diagHom}_H(Q_\bullet(C, M), Q_\bullet(A, M)) \quad \text{and} \quad \alpha_\bullet: \text{Cyc}_\bullet(A) \rightarrow \text{diagHom}_k(C_\bullet(C, M), C_\bullet(A, M))$$

Proof. The first part of the statement follows from Lemma 2.6. The second part follows from the fact that $k \otimes_H (\cdot)$ is a functor from the category of left H -modules to the category of k -modules. \square

Theorem 2.8. *The equivariant action of an H -module coalgebra C on an H -module algebra A induces a pairing of the form*

$$\smile: HC_{\text{Hopf}}^p(C, M) \otimes HC_{\text{Hopf}}^q(A, M) \rightarrow HC^{p+q}(A)$$

for any $p, q \geq 0$.

Proof. First, we observe that

$$HC_{\text{Hopf}}^p(C, M) := \text{Ext}_\Lambda^p(k_\bullet, C_\bullet(C, M)) \quad \text{and} \quad HC_{\text{Hopf}}^q(A, M) := \text{Ext}_\Lambda^q(C_\bullet(A, M), k_\bullet^\vee)$$

We will use the Yoneda interpretation of Ext-groups [24] as developed in [20, Chapter III]. Our approach in part is inspired by the use of Yoneda Ext-groups in [21]. In this approach, one can represent any Hopf-cyclic cohomology class $\xi \in HC_{\text{Hopf}}^p(k_\bullet, C_\bullet(C, M))$ and $\nu \in HC_{\text{Hopf}}^q(C_\bullet(A, M), k_\bullet^\vee)$ by exact sequences of (co)cyclic k -modules

$$\begin{aligned} \xi : 0 \leftarrow k_\bullet \leftarrow Y_\bullet^1 \leftarrow \cdots \leftarrow Y_\bullet^p \leftarrow C_\bullet(C, M) \leftarrow 0 \\ \nu : 0 \leftarrow C_\bullet(A, M) \leftarrow Z_\bullet^1 \leftarrow \cdots \leftarrow Z_\bullet^q \leftarrow k_\bullet^\vee \leftarrow 0 \end{aligned}$$

Since k is a field, the functor $\text{diagHom}_k(\cdot, C_\bullet(A, M))$ from the category of cocyclic k -modules to the category of cyclic k -modules is exact. Hence we get an exact sequence of the form

$$\begin{aligned} \text{diagHom}_k(\xi, C_\bullet(A, M)) : 0 \leftarrow \text{diagHom}_k(C_\bullet(C, M), C_\bullet(A, M)) \leftarrow \text{diagHom}_k(Y_\bullet^p, C_\bullet(A, M)) \leftarrow \cdots \\ \leftarrow \text{diagHom}_k(Y_\bullet^1, C_\bullet(A, M)) \leftarrow C_\bullet(A, M) \leftarrow 0 \end{aligned}$$

after observing the fact that $\text{diagHom}_k(k_\bullet, C_\bullet(A, M)) \cong C_\bullet(A, M)$ as cyclic k -modules. Now splice the sequences $\text{diagHom}_k(\xi, C_\bullet(A, M))$ and ν to get a class in $\text{Ext}_\Lambda^{p+q}(\text{diagHom}_k(C_\bullet(C, M), C_\bullet(A, M)), k_\bullet^\vee)$. However, we also have a morphism of cyclic k -modules α_\bullet constructed in Proposition 2.7. The result follows from the corresponding morphism of Ext-modules

$$\text{Ext}_\Lambda^{p+q}(\alpha_\bullet, k_\bullet^\vee) : \text{Ext}_\Lambda^{p+q}(\text{diagHom}_k(C_\bullet(C, M), C_\bullet(A, M)), k_\bullet^\vee) \rightarrow \text{Ext}_\Lambda^{p+q}(\text{Cyc}_\bullet(A), k_\bullet^\vee)$$

and then observing that $HC^*(A) = \text{Ext}_\Lambda^*(\text{Cyc}_\bullet(A), k_\bullet^\vee)$. \square

If one wishes to write a formula for this pairing, one can write

$$\xi \smile \nu := \text{Ext}_\Lambda^{p+q}(\alpha_\bullet, k_\bullet^\vee) (\text{diagHom}_k(\xi, C_\bullet(A, M)) \circ \nu)$$

for any $\xi \in HC_{\text{Hopf}}^p(C, M)$ and $\nu \in HC_{\text{Hopf}}^q(A, M)$ where \circ denotes the Yoneda product in bivariant cohomology written in the opposite order, i.e.

$$\circ : \text{Ext}_\Lambda^p(X_\bullet, Y_\bullet) \otimes \text{Ext}_\Lambda^q(Y_\bullet, Z_\bullet) \rightarrow \text{Ext}_\Lambda^{p+q}(X_\bullet, Z_\bullet)$$

However, there is one other pairing in the same setting. One can define this second pairing by the formula

$$\nu \smile \xi := \text{Ext}_\Lambda^{p+q}(\alpha_\bullet, k_\bullet^\vee) (\text{diagHom}_k(C_\bullet(C, M), \nu) \circ \text{diagHom}_k(\xi, k_\bullet^\vee))$$

Below, we give an alternative construction for these pairings we gave above and prove that they actually are the same up to a sign.

Proposition 2.9. *Assume that an H -module coalgebra C acts on an H -module algebra A equivariantly. Then for any $\xi \in HC_{\text{Hopf}}^p(C, M)$ and $\nu \in HC_{\text{Hopf}}^q(A, M)$ one has $\xi \smile \nu = (-1)^{pq} \nu \smile \xi$.*

Proof. Since the bi-functor $\text{diagHom}_k(\cdot, \cdot)$ is exact in both variables, one has well-defined morphisms of the form

$$\text{diagHom}_k(Z_\bullet, \cdot) : \text{Ext}_\Lambda^p(X_\bullet, Y_\bullet) \rightarrow \text{Ext}_\Lambda^p(\text{diagHom}_k(Z_\bullet, X_\bullet), \text{diagHom}_k(Z_\bullet, Y_\bullet))$$

and

$$\text{diagHom}_k(\cdot, Z_\bullet) : \text{Ext}_\Lambda^p(X_\bullet, Y_\bullet) \rightarrow \text{Ext}_\Lambda^p(\text{diagHom}_k(Y_\bullet, Z_\bullet), \text{diagHom}_k(X_\bullet, Z_\bullet))$$

for any cocyclic modules X_\bullet, Y_\bullet and cyclic module Z_\bullet . Since $\xi \in HC_{\text{Hopf}}^p(C, M) := \text{Ext}_\Lambda^p(k_\bullet, C_\bullet(C, M))$ and $\nu \in HC_{\text{Hopf}}^q(A, M) := \text{Ext}_\Lambda^q(C_\bullet(A, M), k_\bullet^\vee)$ one has well-defined elements

$$\zeta_1 := \text{diagHom}_k(\xi, C_\bullet(A, M)) \circ \nu \in \text{Ext}_\Lambda^{p+q}(\text{diagHom}(C_\bullet(C, M), C_\bullet(A, M)), k_\bullet^\vee)$$

and

$$\zeta_2 := \text{diagHom}_k(C_\bullet(C, M), \nu) \circ \text{diagHom}_k(\xi, k_\bullet^\vee) \in \text{Ext}_\Lambda^{p+q}(\text{diagHom}(C_\bullet(C, M), C_\bullet(A, M)), k_\bullet^\vee)$$

Here we use the opposite composition notation as before. After observing the fact that $\nu = \text{diagHom}_k(k_\bullet, \nu)$, the proof that one has $\zeta_1 = (-1)^{pq}\zeta_2$ in $\text{Ext}_\Lambda^{p+q}(\text{diagHom}_k(C_\bullet(C, M), C_\bullet(A, M)), k_\bullet^\vee)$, reduces to proving that the bifunctor $\text{diagHom}_k(\cdot, \cdot)$ satisfies the following property

$$\text{diagHom}_k(a_\bullet, Y'_\bullet) \circ \text{diagHom}_k(X'_\bullet, b_\bullet) = \text{diagHom}_k(X'_\bullet, b_\bullet) \circ \text{diagHom}_k(a_\bullet, Y_\bullet)$$

for any morphism of cocyclic modules $a_\bullet: X'_\bullet \rightarrow X_\bullet$ and morphism of cyclic modules $b_\bullet: Y_\bullet \rightarrow Y'_\bullet$ which is obvious. The result follows after observing

$$\xi \smile \nu := \text{Ext}_\Lambda^{p+q}(\alpha_\bullet, k_\bullet^\vee)(\zeta_1) = \text{Ext}_\Lambda^{p+q}(\alpha_\bullet, k_\bullet^\vee)((-1)^{pq}\zeta_2) =: (-1)^{pq}\nu \smile \xi$$

where α_\bullet is the morphism of cyclic modules we constructed in Proposition 2.4. \square

Theorem 2.10. *The Connes-Moscovici characteristic map $HC_{\text{Hopf}}^p(H, k_{(\sigma, \delta)}) \rightarrow HC^p(A)$ defined in [3, Section VIII, Proposition 1] agrees with the pairing we defined in Theorem 2.8 for $C = H$, $M = k_{(\sigma, \delta)}$ and $q = 0$. Here $k_{(\sigma, \delta)}$ denotes the 1-dimensional anti-Yetter-Drinfeld module of the module pair in involution (σ, δ) .*

Proof. Since $k_{(\sigma, \delta)}$ is a stable anti-Yetter-Drinfeld module (an SAYD module in short), by [15] we know that $C_\bullet(H, k_{(\sigma, \delta)})$ is isomorphic to $k \otimes_H T_\bullet(H, k_{(\sigma, \delta)})$. Therefore, one can identify $C_\bullet(H, k_{(\sigma, \delta)})$ as the graded k -submodule of $T_\bullet(H, k_{(\sigma, \delta)})$ which consists of elements the form $\sum_i (1 \otimes h_i^1 \otimes \cdots \otimes h_i^n)$. The Connes-Moscovici characteristic is defined with the help of an invariant trace τ on A which satisfies the following condition

$$\tau(h(a)a') = \tau(aS(h_{(1)})(a')\delta(h_{(2)})) \quad \text{and} \quad \tau(h(a)) = \varepsilon(h)\tau(a)$$

for any $a, a' \in A$ and $h \in H$. This is equivalent to $\tau \in \text{Hom}_k(A \otimes k_{(\sigma, \delta)}, k)$ being a cyclic cochain in degree 0 for the cyclic module $C_\bullet(A, k_{(\sigma, \delta)})$. The characteristic map is defined as

$$\gamma_\bullet(1 \otimes h^1 \otimes \cdots \otimes h^n)(a_0 \otimes \cdots \otimes a_n) = \tau(a_0 h^1(a_1) \cdots h^n(a_n))$$

for any $1 \otimes h^1 \otimes \cdots \otimes h^n \in C_n(H, k_{(\sigma, \delta)})$ and $a_0 \otimes \cdots \otimes a_n \in \text{Cyc}_n(A)$. Now observe that one can write γ_\bullet as a composition $\gamma_\bullet = \tau \circ \alpha_\bullet$ where α_\bullet is defined in the proof of Proposition 2.4. This means γ_\bullet is the morphism

$$\text{Hom}_k(\alpha_\bullet, k): \text{Hom}_k(\text{diagHom}_k(C_\bullet(H, M), C_\bullet(A, M)), k) \rightarrow \text{Hom}_k(\text{Cyc}_\bullet(A), k)$$

where $M = k_{(\sigma, \delta)}$. Then $\text{Hom}_k(\alpha_\bullet, k)$ induces the morphism $\text{Ext}_\Lambda^*(\alpha_\bullet, k_\bullet^\vee)$ on cohomology which is used in the proof of Theorem 2.8 to define the pairing. The result follows. \square

One can extend the pairing we defined above to the equivariant cyclic cohomology groups as follows:

Theorem 2.11. *The equivariant action of an H -module coalgebra C on an H -module algebra A induces a pairing of the form*

$$\smile: HC_H^p(C, M) \otimes HC_H^q(A, M) \rightarrow \bigoplus_{r=0}^{p+q} HC^{p+q-r}(A) \otimes \text{Ext}_H^r(k, k)$$

for any $p, q \geq 0$.

Proof. First observe that the morphism of cyclic modules α_\bullet defined in Proposition 2.4 can also be considered as a morphism of cyclic H -modules of the form $\alpha_\bullet: \text{Cyc}_\bullet(A) \rightarrow \text{diagHom}_k(Q_\bullet(C, M), Q_\bullet(A, M))$. Here $\text{Cyc}_\bullet(A)$ is considered as a trivial H -module and $\text{diagHom}_k(Q_\bullet(C, M), Q_\bullet(A, M))$ has the following H -module structure

$$(hf)(\Psi) := h_{(1)} \cdot f(S(h_{(2)}) \cdot \Psi)$$

for any $f \in \text{diagHom}_k(Q_\bullet(C, M), Q_\bullet(A, M))$, $h \in H$ and $\Psi \in Q_\bullet(C, M)$. Given two cohomology classes

$$\begin{aligned} \mu : 0 \leftarrow k_\bullet \leftarrow U_\bullet^1 \leftarrow \cdots \leftarrow U_\bullet^p \leftarrow Q_\bullet(C, M) \leftarrow 0 \\ \nu : 0 \leftarrow Q_\bullet(A, M) \leftarrow V_\bullet^1 \leftarrow \cdots \leftarrow V_\bullet^q \leftarrow k_\bullet^\vee \leftarrow 0 \end{aligned}$$

in $HC_H^p(C, M) := \text{Ext}_{H[\Lambda]}^p(k_\bullet, Q_\bullet(C, M))$ and in $HC_H^p(A, M) := \text{Ext}_{H[\Lambda]}^p(Q_\bullet(A, M), k_\bullet)$ respectively, we consider the exact sequence of cyclic H -modules

$$\begin{aligned} \text{diagHom}_k(\mu, Q_\bullet(A, M)) : 0 \leftarrow \text{diagHom}_k(Q_\bullet(C, M), Q_\bullet(A, M)) \leftarrow \text{diagHom}_k(U_\bullet^p, Q_\bullet(A, M)) \leftarrow \\ \cdots \leftarrow \text{diagHom}_k(U_\bullet^1, Q_\bullet(A, M)) \leftarrow Q_\bullet(A, M) \leftarrow 0 \end{aligned}$$

We define a class in $\text{Ext}_{H[\Lambda]}^{p+q}(\text{diagHom}_k(Q_\bullet(C, M), Q_\bullet(A, M)), k_\bullet^\vee)$ by splicing $\text{diagHom}_k(\mu, Q_\bullet(A, M))$ and ν at $Q_\bullet(A, M)$. Then we define $\mu \smile \nu \in \text{Ext}_{H[\Lambda]}^{p+q}(\text{Cyc}_\bullet(A), k_\bullet^\vee)$ by using the morphism of cyclic H -modules α_\bullet . However, the H -module structure on $\text{Cyc}_\bullet(A)$ is trivial. Therefore, using the first spectral sequence constructed in [16, Proposition 3.5] we obtain

$$\text{Ext}_{H[\Lambda]}^{p+q}(\text{Cyc}_\bullet(A), k_\bullet^\vee) \cong \bigoplus_{r=0}^{p+q} HC^{p+q-r}(A) \otimes \text{Hom}_k(\text{Tor}_r^H(k, k), k)$$

However, since k is a field we have $\text{Ext}_H^r(U, k) \cong \text{Hom}_k(\text{Tor}_r^H(U, k), k)$ for any $r \geq 0$ and any left H -module U , in particular $U = k$. \square

3. PRODUCT (CO)ALGEBRAS

We will change the notation to distinguish Hopf-cyclic cohomology and equivariant cyclic cohomology of H - and $H \otimes H$ -module algebras. We will use $Q_\bullet(H; A, M)$ to denote the cyclic H -module associated with the H -module algebra A with coefficients in M . Also, we will use $HC_{\text{Hopf}}^*(H; A, M)$ to denote the Hopf-cyclic cohomology of an H -module algebra A with coefficients in M .

Proposition 3.1. *Let A, A' be two H -module algebras and let M and M' be two H -module/comodules. Then there is an external product structure on the equivariant cyclic cohomology groups*

$$HC_H^p(A, M) \otimes HC_H^q(A', M') \rightarrow HC_{H \otimes H}^{p+q}(A \otimes A', M \otimes M')$$

and Hopf-cyclic cohomology groups

$$HC_{\text{Hopf}}^p(H; A, M) \otimes HC_{\text{Hopf}}^q(H; A', M') \rightarrow HC_{\text{Hopf}}^{p+q}(H \otimes H; A \otimes A', M \otimes M')$$

for any $p, q \geq 0$.

Proof. Any cohomology class $\nu \in HC_H^p(A, M)$ and $\nu' \in HC_H^q(A', M')$ can be represented by two exact sequences of cyclic H -modules of the form $\nu : 0 \leftarrow Q_\bullet(A, M) \leftarrow X_\bullet^1 \leftarrow \cdots \leftarrow X_\bullet^p \leftarrow k_\bullet^\vee \leftarrow 0$ and $\nu' : 0 \leftarrow Q_\bullet(A', M') \leftarrow Y_\bullet^1 \leftarrow \cdots \leftarrow Y_\bullet^q \leftarrow k_\bullet^\vee \leftarrow 0$. Now we define the external product $\nu \times \nu'$ by the composition

$$\begin{aligned} 0 \leftarrow \text{diag}(Q_\bullet(H; A, M) \otimes_k Q_\bullet(H; A', M')) \leftarrow \text{diag}(X_\bullet^1 \otimes_k Q_\bullet(H; A', M')) \leftarrow \cdots \\ \leftarrow \text{diag}(X_\bullet^p \otimes_k Q_\bullet(H; A', M')) \leftarrow Y_\bullet^1 \leftarrow \cdots \leftarrow Y_\bullet^q \leftarrow k_\bullet^\vee \leftarrow 0 \end{aligned}$$

which is an exact sequence of cyclic $H \otimes H$ -modules. Here diag denotes the diagonal cyclic structure. Therefore, $\text{diag}(U_\bullet \otimes V_\bullet)$ is a cyclic $H \otimes H$ -module whenever U_\bullet and V_\bullet are cyclic H -modules. Now observe that there is a natural epimorphism from $Q_\bullet(H \otimes H; A \otimes A', M \otimes M')$ onto $\text{diag}(Q_\bullet(H; A, M) \otimes Q_\bullet(H; A', M'))$. This proves our first assertion. The proof of our second assertion is similar. \square

Given two left H -comodules M and M' , we define

$$M \square^H M' := \left\{ \sum_i m_i \otimes m'_i \mid \sum_i m_{i,(-1)} \otimes m_{i,(0)} \otimes m'_i = \sum_i m'_{i,(-1)} \otimes m_i \otimes m'_{i,(0)} \right\}$$

Theorem 3.2. *Let A, A' be two H -module algebras and let M and M' be two H -module/comodules. Assume H is cocommutative Hopf algebra and $M \square^H M'$ is an H submodule of $M \otimes M'$. One has pairings of the form*

$$HC_H^p(A, M) \otimes HC_H^q(A', M') \rightarrow HC_H^{p+q}(A \otimes A', M \square^H M')$$

and

$$HC_{\text{Hopf}}^p(H; A, M) \otimes HC_{\text{Hopf}}^q(H; A', M') \rightarrow HC_{\text{Hopf}}^{p+q}(H; A \otimes A', M \square^H M')$$

for any $p, q \geq 0$.

Now assume Z is a H -comodule coalgebra, i.e. Z is an H -comodule and a coalgebra such that the following compatibility condition is satisfied

$$(3.1) \quad z_{[-1]} \otimes z_{[0](1)} \otimes z_{[0](2)} = z_{(1)[-1]} z_{(2)[-1]} \otimes z_{(1)[0]} \otimes z_{(2)[0]}$$

where the H -comodule structure $\lambda: Z \rightarrow H \otimes Z$ is denoted by $z_{[-1]} \otimes z_{[0]}$ and the comultiplication $\Delta: Z \rightarrow Z \otimes Z$ is denoted by $z_{(1)} \otimes z_{(2)}$ for any $z \in Z$.

Lemma 3.3. *Assume H is commutative. If Z and Z' are two arbitrary H -comodule coalgebras then their product $Z \otimes Z'$ has an H -comodule coalgebra structure.*

Definition 3.4. For a H -comodule coalgebra Z and a stable H -module/comodule M , we define the Hopf-cyclic module $C_\bullet(Z, M)$ associated with the pair (Z, M) as follows: on the graded k -module level we let $C_n(Z, M) := \text{Hom}_{H\text{-comod}}(Z^{\otimes n+1}, M)$ where we view $Z^{\otimes n+1}$ as an H -comodule via the diagonal coaction, i.e.

$$(z^0 \otimes \cdots \otimes z^n)_{[-1]} \otimes (z^0 \otimes \cdots \otimes z^n)_{[0]} := z_{[-1]}^0 \cdots z_{[-1]}^n \otimes (z_{[0]}^0 \otimes \cdots \otimes z_{[0]}^n)$$

for any $z^i \in Z$. The cyclic structure morphisms are defined by

$$\begin{aligned} (\partial_0 f)(z^0 \otimes \cdots \otimes z^{n-1}) &:= f(z_{(1)}^0 \otimes z_{(2)}^0 \otimes z^1 \otimes \cdots \otimes z^{n-1}) \\ (\sigma_0 f)(z^0 \otimes \cdots \otimes z^{n+1}) &:= \varepsilon(z^1) f(z^0 \otimes z^2 \otimes \cdots \otimes z^{n+1}) \\ (\tau_n f)(z^0 \otimes \cdots \otimes z^n) &:= z_{[-1]}^0 f(z^1 \otimes \cdots \otimes z^n \otimes z_{[0]}^0) \end{aligned}$$

for any $f \in C_n(Z, M)$ and $z^i \in Z$. Then we set $\partial_j := \tau_{n-1}^j \partial_0 \tau_n^{-j}$ and $\sigma_i := \tau_{n+1}^i \sigma_0 \tau_n^{-i}$ for $0 \leq j \leq n$ and $0 \leq i \leq n$. The cyclic cohomology of this cyclic H -module will be denoted by $HC_{\text{Hopf}}^*(Z, M)$. The cyclic cohomology of the dual cocyclic object $C_\bullet(Z, M)^\vee$ will be denoted by $HC_{\text{Hopf}}^{\vee,*}(Z, M)$.

Theorem 3.5. *Assume Z and Z' are arbitrary H -comodule coalgebras, M and M' are arbitrary H -module/comodules. If H is commutative and M and M' are symmetric H -modules then there is a pairing of the form*

$$HC_{\text{Hopf}}^{\vee,p}(Z, M) \otimes HC_{\text{Hopf}}^{\vee,q}(Z', M') \rightarrow HC_{\text{Hopf}}^{\vee,p+q}(Z \otimes Z', M \otimes_H M')$$

for any $p, q \geq 0$.

Proof. There is a well-defined morphism of cocyclic k -modules of the form

$$*: \text{diag}(C_\bullet(Z, M) \otimes C_\bullet(Z', M')) \rightarrow C_\bullet(Z \otimes Z', M \otimes_H M')$$

given by the formula

$$(f * f')((x^0, y^0) \otimes \cdots \otimes (x^n, y^n)) := f(x^0 \otimes \cdots \otimes x^n) \otimes_H f'(y^0 \otimes \cdots \otimes y^n)$$

for any $n \geq 0$, $x^i \in Z$ and $y^i \in Z'$. It is easy to see that $*$ is a morphism of simplicial modules. We see that

$$\begin{aligned} (\tau_n f) * (\tau_n f')((x^0, y^0) \otimes \cdots \otimes (x^n, y^n)) &= (\tau_n f)(x^0 \otimes \cdots \otimes x^n) \otimes_H (\tau_n f')(y^0 \otimes \cdots \otimes y^n) \\ &= x_{[-1]}^0 f(x^1 \otimes \cdots \otimes x^n \otimes x_{[-1]}^0) \otimes_H y_{[-1]}^0 f(y^1 \otimes \cdots \otimes y^n \otimes y_{[-1]}^0) \\ &= x_{[-1]}^0 y_{[-1]}^0 f(x^1 \otimes \cdots \otimes x^n \otimes x_{[-1]}^0) \otimes_H f(y^1 \otimes \cdots \otimes y^n \otimes y_{[-1]}^0) \\ &= (\tau_n(f * f'))((x^0, y^0) \otimes \cdots \otimes (x^n, y^n)) \end{aligned}$$

since M is symmetric, as we wanted to show. Now take two exact sequences $\nu: 0 \leftarrow k_\bullet^\vee \leftarrow U_\bullet^1 \leftarrow \cdots \leftarrow U_\bullet^p \leftarrow C_\bullet(Z, M) \leftarrow 0$ and $\nu': 0 \leftarrow k_\bullet^\vee \leftarrow V_\bullet^1 \leftarrow \cdots \leftarrow V_\bullet^q \leftarrow C_\bullet(Z', M') \leftarrow 0$ representing two cyclic cohomology class in $HC_{\text{Hopf}}^{\vee, p}(Z, M)$ and $HC_{\text{Hopf}}^{\vee, q}(Z', M')$ respectively. Consider the exact sequence

$$\begin{aligned} 0 \leftarrow k_\bullet^\vee \leftarrow U_\bullet^1 \leftarrow \cdots \leftarrow U_\bullet^p \leftarrow \text{diag}(C_\bullet(Z, M) \otimes V_\bullet^1) \leftarrow \cdots \\ \leftarrow \text{diag}(C_\bullet(Z, M) \otimes V_\bullet^q) \leftarrow \text{diag}(C_\bullet(Z, M) \otimes C_\bullet(Z', M')) \leftarrow 0 \end{aligned}$$

which represents a class in $\text{Ext}_\Lambda^{p+q}(k_\bullet^\vee, \text{diag}(C_\bullet(Z, M) \otimes C_\bullet(Z', M')))$. Using the morphism $*$ we defined above we get a class in $\text{Ext}_\Lambda^{p+q}(k_\bullet^\vee, C_\bullet(Z \otimes Z', M \otimes_H M'))$. The result follows after observing the fact that

$$\text{Ext}_\Lambda^*(k_\bullet^\vee, X_\bullet) \cong HC^{\vee, p}(X_\bullet)$$

for any cyclic k -module X_\bullet . □

4. CROSSED PRODUCT (CO)ALGEBRAS

In this section, we will assume A is an H -module algebra and B is an H -comodule algebra. M will denote an arbitrary (left-left) stable anti-Yetter-Drinfeld (SAYD) module [9], i.e. M satisfies

$$m_{(-1)}m_{(0)} = m \quad \text{and} \quad (hm)_{(-1)} \otimes (hm)_{(0)} = h_{(1)}m_{(-1)}S^{-1}(h_{(3)}) \otimes h_{(2)}m_{(0)}$$

for any $m \in M$ and $h \in H$.

Definition 4.1. We construct a new algebra $A \rtimes B$ which is defined as $A \otimes B$ on the k -module level. The multiplication structure is defined by the formula

$$(a, b)(a', b') := (a(b_{(-1)}a'), b_{(0)}b')$$

for $(a, b), (a', b') \in A \rtimes B$.

Recall the following definition from [8].

Definition 4.2. The Hopf-cocyclic k -module $C_\bullet(B, M)$ associated with the pair (B, M) is defined by $C_n(B, M) = \text{Hom}_{H\text{-comod}}(B^{\otimes n+1}, M)$ on the graded module level for any $n \geq 0$. This means $f: B^{\otimes n+1} \rightarrow M$ is in $C_n(B, M)$ if and only if

$$(f(b^0 \otimes \cdots \otimes b^n))_{(-1)} \otimes (f(b^0 \otimes \cdots \otimes b^n))_{(0)} = b_{(-1)}^0 \cdots b_{(-1)}^n \otimes f(b_{(0)}^0 \otimes \cdots \otimes b_{(0)}^n)$$

for any $b^i \in B$ for $i = 0, \dots, n$. We let

$$\begin{aligned} (\partial_0 f)(b^0 \otimes \cdots \otimes b^{n+1}) &:= f(b^0 b^1 \otimes b^2 \otimes \cdots \otimes b^{n+1}) \\ (\sigma_0 f)(b^0 \otimes \cdots \otimes b^{n-1}) &:= f(b^0 \otimes 1_B \otimes b^1 \otimes \cdots \otimes b^{n-1}) \\ (\tau_n f)(b^0 \otimes \cdots \otimes b^n) &:= S(b_{(-1)}^n) f(b_{(0)}^n \otimes b^1 \otimes \cdots \otimes b^{n-1}) \end{aligned}$$

Then we let define $\partial_j := \tau_{n+1}^{-j} \partial_0 \tau_n^j$ and $\sigma_i := \tau_{n-1}^{-j} \sigma_0 \tau_n^j$ for $0 \leq j \leq n+1$ and $0 \leq i \leq n$. Note that since M is stable, $\tau_n^{n+1} = id$ for any $n \geq 0$.

Proposition 4.3. *Let us define*

$$\begin{aligned} \beta_n((a_0, b^0) \otimes \cdots \otimes (a_n, b^n))(f) \\ := a_0 \otimes b_{(-n)}^0 a_1 \otimes b_{(-n+1)}^0 b_{(-n+1)}^1 a_2 \otimes \cdots \otimes b_{(-1)}^0 \cdots b_{(-1)}^{n-1} a_n \otimes f(b_{(0)}^0 \otimes \cdots \otimes b_{(0)}^{n-1} \otimes b^n) \end{aligned}$$

for any $n \geq 0$, $(a_i, b^i) \in A \rtimes B$ and $f \in C_n(B, M)$. Then β_\bullet defines a morphism of cyclic modules of the form

$$\beta_\bullet: Cyc_\bullet(A \rtimes B) \rightarrow diagHom_k(C_\bullet(B, M), C_\bullet(A, M))$$

Proof. We see that

$$\begin{aligned} \beta_n(\partial_0((a_0, b^0) \otimes \cdots \otimes (a_n, b^n)))(f) \\ = \beta_n((a_0 b_{(-1)}^0(a_1), b_{(0)}^0 b^1) \otimes (a_2, b^2) \cdots \otimes (a_n, b^n)) \\ = a_0 b_{(-n)}^0(a_1) \otimes b_{(-n+1)}^0 b_{(-n+1)}^1 a_2 \otimes \cdots \otimes b_{(-1)}^0 \cdots b_{(-1)}^{n-1} a_n \otimes f(b_{(0)}^0 b_{(0)}^1 \otimes b_{(0)}^2 \otimes \cdots \otimes b_{(0)}^{n-1} \otimes b^n) \\ = \partial_0(a_0 \otimes b_{(-n)}^0 a_1 \otimes b_{(-n+1)}^0 b_{(-n+1)}^1 a_2 \otimes \cdots \otimes b_{(-1)}^0 \cdots b_{(-1)}^{n-1} a_n) \otimes (\partial_0 f)(b_{(0)}^0 \otimes \cdots \otimes b_{(0)}^{n-1} \otimes b^n) \\ \beta_n(\sigma_0((a_0, b^0) \otimes \cdots \otimes (a_n, b^n)))(f) \\ = \beta_n((a_0, b^0) \otimes (1, 1) \otimes \cdots \otimes (a_n, b^n))(f) \\ = a_0 \otimes 1 \otimes b_{(-n)}^0 a_1 \otimes b_{(-n+1)}^0 b_{(-n+1)}^1 a_2 \otimes \cdots \otimes b_{(-1)}^0 \cdots b_{(-1)}^{n-1} a_n \otimes f(b_{(0)}^0 \otimes 1 \otimes b_{(0)}^1 \otimes \cdots \otimes b_{(0)}^{n-1} \otimes b^n) \\ = \sigma_0(a_0 \otimes b_{(-n)}^0 a_1 \otimes b_{(-n+1)}^0 b_{(-n+1)}^1 a_2 \otimes \cdots \otimes b_{(-1)}^0 \cdots b_{(-1)}^{n-1} a_n) \otimes (\sigma_0 f)(b_{(0)}^0 \otimes \cdots \otimes b_{(0)}^{n-1} \otimes b^n) \end{aligned}$$

and finally we obtain

$$\begin{aligned} \beta_n(\tau_n((a_0, b^0) \otimes \cdots \otimes (a_n, b^n)))(f) \\ = \beta_n((a_n, b^n) \otimes (a_0, b^0) \otimes \cdots \otimes (a_{n-1}, b^{n-1})) \\ = a_n \otimes b_{(-n)}^0 a_0 \otimes b_{(-n+1)}^0 b_{(-n+1)}^1 a_1 \otimes \cdots \otimes b_{(-1)}^0 \cdots b_{(-1)}^{n-2} a_{n-1} \otimes f(b_{(0)}^0 \otimes b_{(0)}^1 \otimes \cdots \otimes b_{(0)}^{n-2} \otimes b^{n-1}) \\ = S^{-1}(b_{(-2)}^0 a_n \otimes a_0 \otimes b_{(-n+1)}^0 a_1 \otimes \cdots \otimes b_{(-1)}^0 \cdots b_{(-1)}^{n-2} a_{n-1} \otimes S(b_{(-1)}^0) f(b_{(0)}^0 \otimes b_{(0)}^1 \otimes \cdots \otimes b_{(0)}^{n-2} \otimes b^{n-1})) \end{aligned}$$

where the last equality follows from the fact that $C_\bullet(A, M)$ is a trivial H -module with respect to the diagonal H -action. Then using the fact that M is a SAYD module and $f \in Hom_{H-comod}(B^{\otimes n+1}, M)$ we get

$$\begin{aligned} \beta_n(\tau_n((a_0, b^0) \otimes \cdots \otimes (a_n, b^n)))(f) \\ = S^{-1}(b_{(-1)(1)}^0) S^{-1}(b_{(0)(-1)}^0 b_{(0)(-1)}^1 \cdots b_{(0)(-1)}^{n-2} b_{(-1)(3)}^0 b_{(-1)}^{n-1} a_n \\ \otimes a_0 \otimes b_{(-n)}^0 a_1 \otimes \cdots \otimes b_{(-2)}^0 \cdots b_{(-2)}^{n-2} a_{n-1} \otimes S(b_{(-1)(2)}^0) f(b_{(0)(0)}^0 \otimes b_{(0)(0)}^1 \otimes \cdots \otimes b_{(0)(0)}^{n-2} \otimes b_{(0)}^{n-1})) \\ = \tau_n(\beta_n((a_0, b_0) \otimes \cdots \otimes (a_n, b_n))(\tau_n f)) \end{aligned}$$

as we wanted to show. \square

Theorem 4.4. *Assume M is a SAYD module over H . If A is an H -module algebra and B is an H -comodule algebra then there is a pairing of the form*

$$\smile: HC_{Hopf}^p(A, M) \otimes HC_{Hopf}^q(B, M) \rightarrow HC^{p+q}(A \rtimes B)$$

for any $p, q \geq 0$.

Proof. Assume we have two exact sequences

$$\begin{aligned} \nu: 0 \leftarrow k_\bullet \leftarrow V_\bullet^1 \leftarrow \cdots \leftarrow V_\bullet^p \leftarrow C_\bullet(B, M) \leftarrow 0 \\ \nu': 0 \leftarrow C_\bullet(A, M) \leftarrow W_\bullet^1 \leftarrow \cdots \leftarrow W_\bullet^q \leftarrow k_\bullet^\vee \leftarrow 0 \end{aligned}$$

representing two cyclic classes $\nu \in HC_{\text{Hopf}}^p(A, M)$ and $\nu' \in HC_{\text{Hopf}}^q(B, M)$. We construct

$$\begin{aligned} \text{diagHom}_k(\nu, C_\bullet(A, M)): 0 \leftarrow \text{diagHom}_k(C_\bullet(C, M), C_\bullet(A, M)) \leftarrow \text{diagHom}_k(V_\bullet^p, C_\bullet(A, M)) \leftarrow \\ \cdots \leftarrow \text{diagHom}_k(V_\bullet^1, C_\bullet(A, M)) \leftarrow C_\bullet(A, M) \leftarrow 0 \end{aligned}$$

and then splice it with ν' at $C_\bullet(A, M)$ to get a class in $HC^{p+q}(\text{diagHom}_k(C_\bullet(B, M), C_\bullet(A, M)))$. Now we use the morphism β_\bullet of cyclic modules we constructed in Proposition 4.3 to get a class in $HC^{p+q}(A \rtimes B)$. \square

Definition 4.5. Given an H -module coalgebra C and an H -comodule coalgebra Z , we define the crossed product coalgebra $Z \ltimes C$ as follows: we let $Z \ltimes C := Z \otimes C$ on the k -module level. The counit on $Z \ltimes C$ is the tensor product of counits on C and Z respectively. The comultiplication structure is defined by the formula

$$(z, c)_{(1)} \otimes (z, c)_{(2)} := (z_{(1)}, z_{(2)[-1]}c_{(1)}) \otimes (z_{(2)[0]}, c_{(2)})$$

for any $(c, z) \in Z \ltimes C$. For coassociativity, one must have

$$\begin{aligned} (z, c)_{(1)(1)} \otimes (z, c)_{(1)(2)} \otimes (z, c)_{(2)} &= (z_{(1)}, z_{(2)[-1]}c_{(1)}) \otimes (z_{(2)[0]}, c_{(2)})_{(1)} \otimes (z_{(2)[0]}, c_{(2)})_{(2)} \\ &= (z_{(1)}, z_{(2)[-1]}c_{(1)}) \otimes (z_{(2)[0](1)}, z_{(2)[0](2)[-1]}c_{(2)}) \otimes (z_{(2)[0](2)[0]}, c_{(3)}) \end{aligned}$$

The compatibility conditions for comodule coalgebras give in Equation (3.1) imply

$$z_{[-1]} \otimes z_{[0](1)} \otimes z_{[0](2)[-1]} \otimes z_{[0](2)[0]} = z_{(1)[-1]}z_{(2)[-1]} \otimes z_{(1)[0]} \otimes z_{(2)[-1]} \otimes z_{(2)[0]}$$

This in turn yields

$$\begin{aligned} (z, c)_{(1)(1)} \otimes (z, c)_{(1)(2)} \otimes (z, c)_{(2)} &= (z_{(1)}, z_{(2)[-1]}z_{(3)[-1]}c_{(1)}) \otimes (z_{(2)[0]}, z_{(3)[-1]}c_{(2)}) \otimes (z_{(3)[0]}, c_{(3)}) \\ &= (z, c)_{(1)} \otimes (z, c)_{(2)(1)} \otimes (z, c)_{(2)(2)} \end{aligned}$$

for any $z \in Z$ and $c \in C$ as we wanted to show.

Proposition 4.6. Assume M is an SAYD module. Let us define ξ_n by the formula

$$\begin{aligned} \xi_n((z^0, c^0) \otimes \cdots \otimes (z^n, c^n))(f) \\ &:= S^{-1}(z_{[-1]}^0 \cdots z_{[-1]}^n)c^0 \otimes S^{-1}(z_{[-2]}^1 \cdots z_{[-2]}^n)c^1 \otimes \cdots \otimes S^{-1}(z_{[-n-1]}^n)c^n \otimes f(z_{[0]}^0 \otimes \cdots \otimes z_{[0]}^n) \\ &= S^{-1}(z_{[-1]}^0 \cdots z_{[-1]}^{n-1})c^0 \otimes S^{-1}(z_{[-2]}^1 \cdots z_{[-2]}^{n-1})c^1 \otimes \cdots \otimes S^{-1}(z_{[-n]}^{n-1})c^{n-1} \otimes c^n \otimes z_{[-1]}^n f(z_{[0]}^0 \otimes \cdots \otimes z_{[0]}^n) \end{aligned}$$

for any $n \geq 0$, $c^i \in C$, $z^i \in C$ and $f \in C_n(Z, M)$. Then ξ_\bullet is a morphism of cocyclic modules of the form

$$\xi_\bullet: \text{Cyc}_\bullet(Z \ltimes C) \rightarrow \text{diagHom}_k(C_\bullet(Z, M), C_\bullet(C, M))$$

Proof. We will prove that ξ_\bullet and the cyclic operators are compatible but we will leave the verification of the fact that ξ_\bullet is compatible with the face and degeneracy maps to the reader. We also observe

$$\begin{aligned} \xi_n(\tau_n((z^0, c^0) \otimes \cdots \otimes (z^n, c^n)))(f) \\ &= \xi_n((z^1, c^1) \otimes \cdots \otimes (z^n, c^n) \otimes (z^0, c^0))(f) \\ &= S^{-1}(z_{[-1]}^1 \cdots z_{[-1]}^n)c^1 \otimes S^{-1}(z_{[-2]}^2 \cdots z_{[-2]}^n)c^2 \otimes \cdots \otimes S^{-1}(z_{[-n]}^n)c^n \otimes c^0 \otimes z_{[-1]}^0 f(z_{[0]}^1 \otimes \cdots \otimes z_{[0]}^n \otimes z_{[0]}^0) \\ &= S^{-1}(z_{[-2]}^1 \cdots z_{[-2]}^n)c^1 \otimes S^{-1}(z_{[-3]}^2 \cdots z_{[-3]}^n)c^2 \otimes \cdots \otimes S^{-1}(z_{[-n-1]}^n)c^n \otimes \\ &\quad z_{[0] [-4]}^0 z_{[0] [-1]}^1 \cdots z_{[0] [-1]}^n z_{[0] [-1]}^0 S^{-1}(z_{[0] [-2]}^0) S^{-1}(z_{[-1]}^0 z_{[-1]}^1 \cdots z_{[-1]}^n)c^0 \otimes z_{[0] [-3]}^0 f(z_{[0] [0]}^1 \otimes \cdots \otimes z_{[0] [0]}^n \otimes z_{[0] [0]}^0) \\ &= \tau_n(\xi_n((z^0, c^0) \otimes \cdots \otimes (z^n, c^n))(\tau_n f)) \end{aligned}$$

for any $n \geq 0$, $z^i \in Z$, $c^i \in C$ and $f \in C_n(Z, M)$. \square

Theorem 4.7. *Fix an SAYD module M over H . For an H -module coalgebra C and an H -comodule coalgebra Z one has a pairing of the form*

$$HC^p(Z \ltimes C) \otimes HC_{\text{Hopf}}^{\vee, q}(Z, M) \rightarrow HC_{\text{Hopf}}^{p+q}(C, M)$$

for any $p, q \geq 0$ where $HC_{\text{Hopf}}^{\vee, *}(Z, M)$ denotes the cyclic cohomology of the dual (cocyclic) module of the cyclic k -module $C_{\bullet}(Z, M)$.

Proof. Any class $[\nu]$ in $HC_{\text{Hopf}}^{\vee, q}(Z, M)$ represented by an exact sequence of the form $\nu: 0 \leftarrow k_{\bullet}^{\vee} \leftarrow X_{\bullet}^1 \leftarrow \dots \leftarrow X_{\bullet}^q \leftarrow C_{\bullet}(Z, M) \leftarrow 0$. Now one can construct a class $\text{diagHom}_k(\nu, C_{\bullet}(C, M))$ in the bivariant cyclic cohomology group $\text{Ext}_{\Lambda}^q(\text{diagHom}_k(C_{\bullet}(Z, M), C_{\bullet}(C, M)), C_{\bullet}(C, M))$ via

$$\begin{aligned} 0 \leftarrow \text{diagHom}_k(C_{\bullet}(Z, M), C_{\bullet}(C, M)) &\leftarrow \text{diagHom}_k(X_{\bullet}^q, C_{\bullet}(C, M)) \leftarrow \dots \\ &\leftarrow \text{diagHom}_k(X_{\bullet}^1, C_{\bullet}(C, M)) \leftarrow C_{\bullet}(C, M) \leftarrow 0 \end{aligned}$$

By using ξ_{\bullet} we defined in Proposition 4.6 we obtain a class in $\text{Ext}_{\Lambda}^q(\text{Cyc}_{\bullet}(Z \ltimes C), C_{\bullet}(C, M))$. In other words, we have a morphism of graded modules of the form

$$HC_{\text{Hopf}}^{\vee, *}(Z, M) \rightarrow \text{Ext}_{\Lambda}^*(\text{Cyc}_{\bullet}(Z \ltimes C), C_{\bullet}(C, M))$$

Now pairing with the cyclic cohomology of the crossed product coalgebra $HC^p(Z \ltimes C) := \text{Ext}_{\Lambda}^p(k_{\bullet}, \text{Cyc}_{\bullet}(Z \ltimes C))$ and using the Yoneda composition we get the desired cup product. \square

5. COMPARISON THEOREM

The first examples of pairings of the form 2.8 between Hopf-cyclic cohomology of a module algebra A and the comodule algebra $C = H$ is the Connes-Moscovici characteristic map (for only $q = 0$) [3] and its extension to differential graded setting by Gorokhovsky [7] for $q = 0, 1$ with periodic cyclic cohomology but only for the 1-dimensional coefficient module $k_{(\sigma, \delta)}$ coming from a modular pair in involution. In [18], Khalkhali and Rangipour defined two cup products for arbitrary module algebras and coalgebras, and their “cup product of the second kind” agrees with Gorokhovsky’s extended characteristic map, and therefore with Connes-Moscovici characteristic map as well, when the coefficient module is $k_{(\sigma, \delta)}$. There are yet other ways of defining cup products in the context of Hopf equivariant Cuntz-Quillen formalism due to Crainic [4] and Nikonov-Sharygin [23] where the former constructs the cup product only for $C = H$ and for the 1-dimensional SAYD module $k_{(\sigma, \delta)}$ while the latter generalizes the construction to arbitrary H -module coalgebras and arbitrary SAYD coefficient modules.

Our approach to products is different than all of the approaches we enumerate above in that all the cup products mentioned above use the theory of abstract cycles and closed graded traces (see [2, p. 183], [19, p. 74] or [18]), and/or the homotopy category of (special) towers of super complexes. Thanks to Quillen [22] we know that the category of (special) towers of super complexes is homotopy equivalent to the category of mixed complexes and the category of S -modules. These homotopy/derived categories are different than the derived category of (co)cyclic modules as we observed in the Introduction. So far, we explicitly used the derived category of (co)cyclic k -modules or H -modules depending on whether we needed the bivariant Hopf or equivariant bivariant cyclic cohomology. An alternative approach would be to develop a similar theory by using the derived category of mixed complexes. Recall that in the construction of various products, we heavily relied on the exact bi-functor $\text{diagHom}_k(\cdot, \cdot)$ which takes a pair of cocyclic and cyclic k -modules as an input and which produces a cyclic k -module. If we were to construct similar products using the derived category of mixed complexes, we need to mimic the same construction.

One can view the category of mixed complexes as the category of differential graded \mathcal{M} -modules where \mathcal{M} is the free graded symmetric algebra of the graded vector space k where the single generator is assumed to have degree 1. We will suggestively use $B \in \mathcal{M}$ to denote this generator. Then the derived category of differential graded \mathcal{M} -modules is the homotopy category of mixed complexes [11].

Lemma 5.1 ([21] Proposition 1.5). *The functor $\mathcal{B}_*: \mathbf{mod}\text{-}\Lambda \rightarrow \mathcal{M}\text{-dgmod}$ which sends a cyclic module to its mixed complex is an exact functor. Therefore it induces natural morphisms of derived bifunctors*

$$\mathrm{Ext}_\Lambda^*(\cdot, \cdot) \rightarrow \mathbf{Ext}_{\mathcal{M}}^*(\cdot, \cdot)$$

where $\mathbf{Ext}_{\mathcal{M}}$ stands for the derived functor of the Hom-bifunctor of the category of differential graded \mathcal{M} -modules.

A bi-differential graded k -module $(X_{*,*}; b_1, b_2)$ is called a mixed double complex if there exists differentials B_1 and B_2 of degree 1 such that $[b_i, b_j] = [b_i, B_j] = [B_i, B_j] = 0$ whenever $i \neq j$ and $b_i B_i + B_i b_i = 0$ for $i = 1, 2$. For any mixed double complex $(X_{*,*}; b_1, b_2, B_1, B_2)$ we will use $Tot_*(X_{*,*})$ to denote the mixed complex over the total complex of the bi-differential graded module $X_{*,*}$ with the degree 1 differential $B = B_1 + B_2$.

Lemma 5.2. *Assume X_\bullet is an arbitrary cocyclic k -module and Y_\bullet is an arbitrary cyclic k -module. Then in the derived category of mixed complexes there is an isomorphism of the form*

$$\eta_*: Tot_* \mathrm{Hom}_k(\mathcal{B}_*(X_\bullet), \mathcal{B}_*(Y_\bullet)) \rightarrow \mathcal{B}_*(\mathrm{diag} \mathrm{Hom}_k(X_\bullet, Y_\bullet))$$

Proof. We know that $\mathrm{Hom}_k(X_\bullet, Y_\bullet)$ is a bi-cyclic k -module. If this bi-cyclic module was a product of two cyclic modules then we could have used [10]. Unless X_n is finite dimensional for any $n \geq 0$, this is not the case. But, we can still use [6, Theorem 3.1] with $T = id$ to get the desired isomorphism η_* . \square

Lemma 5.3. *Let M be an arbitrary H -module/comodule. Assume A is an H -module algebra, B is an H -comodule algebra and C is a H -module coalgebra. Then there are morphisms of mixed complexes of the form*

$$\begin{aligned} Tot_* \mathrm{Hom}_k(\mathcal{B}_*(C_\bullet(C, M)), \mathcal{B}_*(C_\bullet(A, M))) &\xrightarrow{\eta_*} \mathcal{B}_*(\mathrm{diag} \mathrm{Hom}_k(C_\bullet(C, M), C_\bullet(A, M))) \xleftarrow{\mathcal{B}_*(\alpha_\bullet)} \mathcal{B}_*(Cyc_\bullet(A)) \\ Tot_* \mathrm{Hom}_k(\mathcal{B}_*(C_\bullet(B, M)), \mathcal{B}_*(C_\bullet(A, M))) &\xrightarrow{\eta_*} \mathcal{B}_*(\mathrm{diag} \mathrm{Hom}_k(C_\bullet(B, M), C_\bullet(A, M))) \xleftarrow{\mathcal{B}_*(\beta_\bullet)} \mathcal{B}_*(Cyc_\bullet(A \rtimes B)) \end{aligned}$$

where on the second row, we require M to be SAYD as well.

Proof. Left-ward arrows come from applying the functor \mathcal{B}_* to α_\bullet we constructed in Proposition 2.4 and Proposition 2.7, and to β_\bullet which is constructed in Proposition 4.3. The right-ward arrows come from Lemma 5.2. \square

Theorem 5.4. *One can define pairings analogous to the pairings we defined in Theorem 2.8, Theorem 2.11 and Theorem 3.2 by using the derived category of mixed complexes instead of the derived category of cyclic modules. However, these pairings defined in the derived category of mixed complexes agree with those defined in the derived category of cyclic modules.*

Proof. The fact that η_* is an isomorphism in the derived category of mixed complexes allows us to use $Tot_* \mathrm{Hom}_k(\mathcal{B}_*(\cdot), \mathcal{B}_*(\cdot))$ as a replacement for $\mathrm{diag} \mathrm{Hom}_k(\cdot, \cdot)$ in the category of mixed complexes. Then Lemma 5.1 gives us a comparison map between these pairings. However, since we are comparing cyclic cohomology of cyclic modules computed via the cyclic double complex and the (b, B) -complex, we see that these groups are the same. \square

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E-mail address: kaygun@math.ohio-state.edu

DEPARTMENT OF MATHEMATICS, THE OHIO STATE UNIVERSITY, COLUMBUS 43210, OHIO USA